# Analysis of Non-Coherent Correlation in DS/BPSK Spread Spectrum Acquisition

Vladan M. Jovanović, *Member, IEEE*, and Elvino S. Sousa, *Member, IEEE*

*Abstract*— **Non-coherent detectors for initial code synchronization (acquisition) of BPSK direct sequence spread spectrum signals on an AWGN channel are analyzed. In addition to the thermal noise, in many applications such detectors are faced with the "self-noise", due to the partial period correlations. Under the random code sequences assumption, in this paper an exact analysis of the non-coherent correlator's detection performance is carried out by using the theory of circularly symmetric random variables. The exact analysis shows that the familiar Gaussian approximation to the distribution function of the code self-noise is justified for all cases of practical interest. Furthermore, the overall detection performance was found to be determined asymptotically by the sum of the thermal and correlator's self-noise. In most cases of practical interest, this asymptotic result provides a very good approximation to the actual detection performance of a non-coherent correlator, improving the approximations devised previously.**

## I. Introduction

Acquisition, or initial coarse synchronization, of direct sequence binary phase shift keying (DS/BPSK) spread spectrum waveforms is usually achieved through non-coherent correlation, as the signal to noise ratio prior to despreading is usually insufficient for the satisfactory performance of practical carrier phase estimators based on tracking loops.

In addition to the additive white Gaussian noise (AWGN), during acquisition the non-coherent BPSK correlator is also confronted with correlator "self-noise", due to the pseudo noise (PN) variations of the detected signal in the absence of any external noise that arises from PN code autocorrelation variations. The situation somewhat resembles that encountered in the multiple access environment, in which there is a single, nonsynchronized, interferering signal.

The earliest analyses of non-coherent correlation were based on the assumption that the self-noise could be neglected (see [1], vol. III, pp. 31–35, and references therein). For early spread spectrum systems this approximation was well justified, as these systems were designed primarily for antijam or low probability of intercept applications, in which the thermal noise or jamming noise would typically be much larger than the correlator self-noise. For modern commercial applications, however, such as in indoor wireless or mobile cellular systems, this assumption is not necessarily true.

Thus far, within the context of spread spectrum acquisition, two different approaches to the analysis of the correlator self-noise have been utilized. The first one is based on the worst case bound on the partial period autocorrelations for the maximal length PN sequences, derived by Hemmati and Schilling [2], and later used several times (e.g.[3]). Besides being valid only for the maximal length family of sequences, this method is in general somewhat pessimistic, and appears to be particularly pessimistic for correlation periods significantly shorter than the code period[1], which seem to be the dominant case of interest in present non-military applications.

The other approach, adopted by Polydoros and Weber in [4], is to model the self-noise as a Gaussian random process. This procedure is based on the intuitively appealing assumption that the partial period autocorrelations are distributed binomially (for additional justification, see [1], vol. I, pp. 289–295), and on the central limit theorem arguments that are applicable for large integration periods. After considerable mathematical manipulation, series representations for the detection and false alarm probabilities were derived, and a simple, although somewhat intuitive, approximation was devised. This approximation suggests that the overall detection performance can be approximated by the results obtained with an "equivalent" noise equal to the sum of the thermal noise and one half of the correlator self-noise.

From similar analyses of the multiple access interference effects upon the coherent correlator, however, it is known that the Gaussian approximation to the binomially distributed correlations might be somewhat questionable for small numbers of interfering signals if one is considering the tails of the distribution, i.e. very low probabilities of erroneous detection [5]. Unlike the situation in coherent correlation for data demodulation, where the case of very low error probabilities is not necessarily interesting if the spread spectrum system incorporates some kind of error correcting coding [6], in spread spectrum synchronization we are quite frequently dealing with comparatively lower error probabilities, especially for the probability of false alarm.

Paper approved by Sorin Davidovici, the Editor for Spread Spectrum of the IEEE Communications Society. Manuscript received: June 17, 1992; revised March 26, 1993. This research was supported by a grant from the Canadian Institute for Telecommunications Research under the NCE program of the Government of Canada. This paper was presented in part at the 16th Biennial Symposium on Communications, Queen's University, May 1992 and at and at the IEEE Military Communications Conference (MILCOM), San Diego, California, October 1992.

V. M. Jovanović is with Bell Mobility Cellular, Etobicoke, Ontario, Canada. E. S. Sousa is with the Department of Electrical and Computer Engineering, University of Toronto, Toronto, Ontario M5S 1A4, Canada.

[1] For instance, for a "typical" correlation period of 100 chips, and assuming a sequence length of only 4000 chips, a somewhat simplified version of the "worst case" bound as derived in [3] would guarantee a partial period "off phase" autocorrelation smaller than 97.5, the "in phase" correlation being equal to 100.

One goal of this paper, therefore, is to verify the validity of the Gaussian approximation to the binomial distribution of the partial period autocorrelations, and the other is to obtain more rigorously some handy approximations for evaluating the performance of the non-coherent correlator.

This paper is organized as follows. In Section II, a brief introduction to the non-coherent correlation problem is presented. The characteristics of the binomial distribution, and the corresponding Gaussian approximation, are discussed in Section III. In Section IV, it is shown that the exact analysis of both models can be carried out by using the theory of circularly symmetric random variables. Finally, asymptotically tight approximations for the detection error probabilities are derived in Section V.

## II. Non-Coherent DS/BPSK Correlation

In a DS/BPSK spread spectrum system the received signal is

$$x(t) = \sqrt{\frac{2E_c}{T_c}} \sum_{k=-\infty}^{\infty} \{d_{int[(k+K)/L]}\ c_{k+K}\ h(t - kT_c - \tau) \cdot \cos(\omega_0 t + \Theta)\} + n(t). \quad (1)$$

where $n(t)$ is a white Gaussian noise process with power spectral density $N_0/2$, $T_c$ is the chip (PN code symbol) duration, $E_c$ is the received energy per chip, $h(\cdot)$ is a rectangular pulse of duration $T_c$, $K$ is the phase of the code-chip sequence, and $\tau$ and $\Theta$ are the unknown chip clock and carrier phases, respectively. The data sequence, $\{d_j\}$, and the PN code sequence, $\{c_j\}$, both assume values from the set $\{+1, -1\}$. $L$ is the number of code-sequence chips per data symbol.

To initiate successful reception, the receiver must estimate the actual phase of the code chip sequence $K$, the code clock phase $\tau$, and the carrier phase $\Theta$; the latter two phases are assumed to be uniformly distributed over $[0, T_c]$ and $[0, 2\pi]$ respectively. Theoretically, the receiver could perform a joint estimation, or separate estimations of $K$, $\tau$ and $\Theta$ in any order. In practice, however, due to the low pre-despreading signal-to-noise ratio, the receiver is forced to estimate the code sequence phase ($K$) followed by $\tau$ and $\Theta$, usually through the use of a delay-lock loop and a Costas loop respectively.

Initial estimation of the code sequence phase, also known as code acquisition, or initial coarse code synchronization, is usually performed through serial-search methods. These are trial-and-error procedures, whereby the correlation of the incoming signal and an arbitrarily phased locally generated PN waveform is performed. If the result shows that these two are not closely correlated, the relative phase of the local signal is readjusted, and the process is repeated until the synchronization detector indicates that the correct code-sequence phase has been found. Typically, this process would result in an estimation error somewhere between one-quarter and one-half of the chip period, depending on the step size. Fine estimation is achieved in the second stage and is referred to as tracking ([1], pp. 153–206).

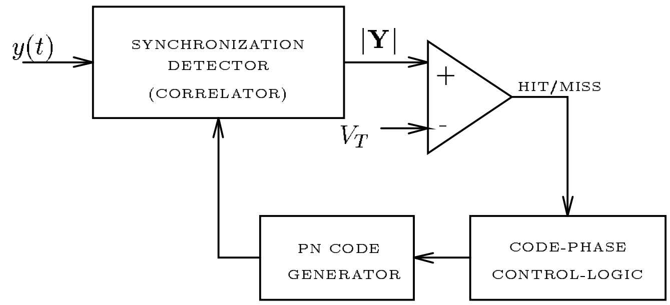


Fig. 1. Serial-search acquisition scheme.

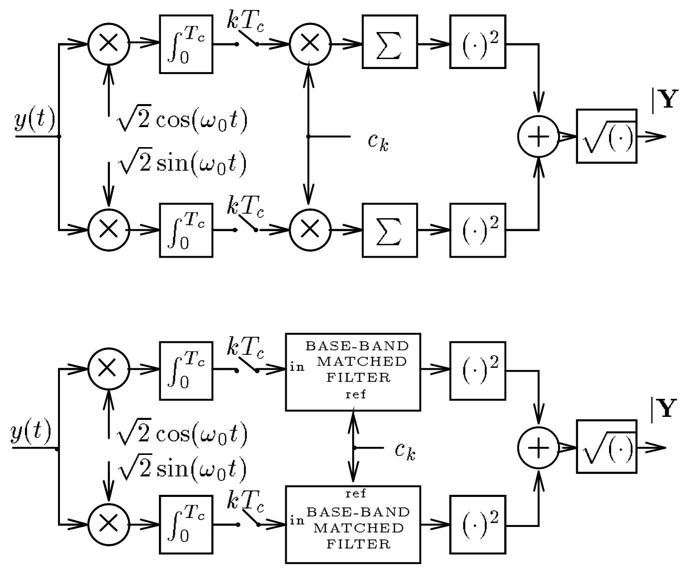


Fig. 2. Non-coherent I&Q correlators: *active*(top) and *passive* (bottom).

Since carrier-phase estimation is performed after the code synchronization, the serial-search correlators in Fig. 1 must be non-coherent. Two such schemes, based on a quadrature I&Q detector and base-band processing, are depicted in Fig. 2. The first is the so called "active" correlator, where the correlation is performed sequentially on a "chip-by-chip" basis, and the second is the "passive" correlator (or matched filter), where the correlation is performed on $M$ chips in parallel. The latter could be realized with the aid of charge-coupled-devices (CCD) [11], or programmable digital FIR filters [12].

We note that analogous schemes exist (yielding the same performance) where the correlation is performed at the carrier (or some intermediate) frequency (RF correlation). "RF passive" correlators are very simple, and "RF" matched filters can be realized, for example, with the aid of SAW devices [10]. For an in-depth survey of the technological and implementation issues, see [13].

The main difference between the two schemes in Fig. 2 reflects the rate of bringing the decisions: in order to correlate $M$ chips, "active" correlators require $MT_c$ seconds, while the "passive" ones can bring such a decision each $T_c$ seconds, thus speeding up the acquisition process by a factor of $M$ (typical value is $M \approx 100$). Apart from that, their performances are equivalent.

For simplicity, in the following we will neglect the data sequence $\{d_j\}$, either because no data is transmitted during the initial acquisition period, or because its effects can be neglected (e.g. data period $LT_c$ is much larger than the integration period $MT_c$, data transition instants are related to the code sequence $\{c_j\}$ in such a manner that the integrations are always performed within the same data symbol, etc.). However, the effects of the data can be easily taken into account following the technique presented in [14].

## III. Correlator Modeling

Neglecting the effect of data modulation, after despreading, integration and envelope detection, the decision variable at the output of the non-coherent correlator is the modulus of the complex *r.v.* $\mathbf{Y}$ defined as

$$\mathbf{Y} = R\, e^{j\Theta} + \mathbf{N}, \tag{2}$$

where $\mathbf{N}$ is a complex, zero-mean Gaussian *r.v.* with *i.i.d.* components with variances given by

$$\sigma_N^2 = \frac{N_0 M T_c}{2}, \tag{3}$$

and $R$ is a real *r.v.*, defined as follows:

$$R = \sqrt{\frac{E_c}{T_c}}\ \left[\tau\, \mathcal{R}_M(K-1, D+1) + (T_c - \tau)\, \mathcal{R}_M(K, D)\right]. \tag{4}$$

In (4), $\mathcal{R}_M(K,D)$ represents the partial auto-correlation

$$\mathcal{R}_M(K, D) = \sum_{k=1}^{M} c_{k+K}\, c_{k+K+D}\,, \tag{5}$$

$D$ being the offset (in number of code chips) between the locally generated and incoming code sequences. It can be either positive or negative.

To evaluate $\mathcal{R}_M(K,D)$ exactly for any given M, partial period cross-correlations for the specific code sequence selected in the system must be determined for each pair $(K,D)$. The most meaningful results would then be obtained by averaging over all $K$ and $D$. Unfortunately, for fairly long codes these computations would be prohibitive because of the time required for processing, and the results would only apply to the particular code sequence chosen.

Another approach, that is frequently used in similar situations [5; 6], is to assume the sequence $\{c_j\}$ to be random[2]. The *r.v.* $\mathcal{R}_M(K,D)$ in that case is distributed binomially, i.e. for $D \neq 0$ its probability-density-function (*p.d.f.*) is

$$\mathcal{P}_{\mathcal{R}_M(K,D)}(r) = 2^{-M} \sum_{n=0}^{M} \binom{M}{n}\, \delta\left[r - (2n - M)\right], \tag{6}$$

[2]Without some "side information" [9], synchronization cannot be acquired if the sequences are really random (implying that the sequence length tends to infinity). In that case "random sequence" means a repetitive sequence of length $N$ with each $c_j (1 \leq j \leq N)$ chosen randomly from $\{+1, -1\}$, and the "average" performance is then taken over the ensemble. With side information, e.g. with precise time references [9], codes can be random in a sense as in [5; 6].

and for $D = 0$

$$\mathcal{P}_{\mathcal{R}_M(K,O)}(r) = \delta\left(r - M\right), \tag{7}$$

where $\delta(\cdot)$ is the Dirac delta-function. From (4) we see that, for the random sequence model, the *p.d.f.* of the *r.v.* $R$ in (2) is a convolution of two binomial densities. In addition, under the random sequence assumption, the partial period correlations $\mathcal{R}_M(K,D)$ do not depend on the code phase $K$, but only on the offset $D$.

We note that the binomial distribution for the partial period correlation $\mathcal{R}_M(K,D)$ in (5) is intuitively expected to hold not only for true random sequences, but for any long pseudo-random sequence satisfying Golomb's three postulates of randomness ([15], see also [1], vol. I, pp. 289–295). Indeed, precise analyses show that this assertion holds for most (although not all) members of the most frequently used classes of PN sequences, such as the maximal-length or Gold sequences, as long as $\log_2 N < M < N$ [16], $N$ being the sequence length.

Further simplifications are possible if the integrations are performed over many chips ($M \gg 1$), which is usually the case in practice. By central-limit-theorem arguments, the distribution of $\mathcal{R}_M(K,D)$ in (5) may be approximated by a Gaussian distribution with mean

$$m_D = \begin{cases} M, & D = 0 \\ 0, & D \neq 0 \end{cases}\,, \tag{8}$$

and variance

$$\sigma_D^2 = \begin{cases} 0, & D = 0 \\ M, & D \neq 0 \end{cases}\,. \tag{9}$$

The random variable $R$ in (4), being a sum of two independent Gaussian *r.v.'s*, is now also Gaussian. From (4) and (5), the mean of $R$ is non-zero only for $D = 0$ or $D = -1$, i.e. when the two codes overlap at least partially; we denote this as *hypothesis* $H_1$. The alternative case (codes not aligned) is denoted as *hypothesis* $H_0$ and corresponds to $R$ having zero mean.

If we let $\Delta$ denote the fractional normalized timing offset between codes; i.e.

$$\Delta = \begin{cases} \frac{\tau}{T_c}, & D \neq -1 \\ 1 - \frac{\tau}{T_c}, & D = -1 \end{cases}\,, \tag{10}$$

then conditioned on hypothesis $H_i\,(i = 0, 1)$, the mean of $R$ is

$$m_i = \begin{cases} \sqrt{E_c T_c}(1 - \Delta) M, & i = 1 \\ 0, & i = 0 \end{cases}\,, \tag{11}$$

and the variance of $R$ is

$$\sigma_i^2 = \begin{cases} E_c T_c \Delta^2 M, & i = 1 \\ E_c T_c (1 - 2\Delta + 2\Delta^2) M, & i = 0 \end{cases}\,. \tag{12}$$

As mentioned earlier, the binomial distribution for the partial period correlations is expected to be applicable not only for true random sequences, but for any long PN sequence likely to be used in practice. The Gaussian approximation to the binomial distribution, on the other hand, is expected to hold very well for $M \gg 1$, at least as long as we are not interested in the tails of the distribution. Since in spread spectrum synchronization we are mostly dealing with low error-probabilities (say $< 10^{-3}$, especially for the probability of the false alarm), the validity of the Gaussian approximation needs to be verified. It appears that this has not yet been done.

## IV. Exact Analysis

We are interested in obtaining the distribution for the modulus $|\mathbf{Y}|$ of the complex *r.v.* $\mathbf{Y}$ given by (2). This problem has been studied extensively in contexts of multipath reception [17], demodulation in the presence of co-channel interference [18], analog FDMA [19], and a host of other problems. These previous analyses were based on the theory of circularly symmetric random vectors. From (2) it is readily seen that the *p.d.f.* of the complex *r.v.* $\mathbf{Y}$ has circular symmetry, so that the results from the theory of two-dimensional circularly symmetric random vectors can be applied.

Several methods have been devised for the analysis of the distribution of $|\mathbf{Y}|$ [17]. The approach that we found to be the most convenient in the present application is a Laguerre polynomial method proposed by Esposito and Wilson [20], and further generalized by Goldman [21]. Goldman derived the following series representation for the *p.d.f.* of $|\mathbf{Y}|$

$$\mathcal{P}_{|\mathbf{Y}|}(y) = \frac{y}{\sigma_N^2}\exp\left(-\frac{y^2}{2\sigma_N^2}\right) \cdot \sum_{k=0}^{\infty}\frac{1}{k!}\left(-\frac{1}{2\sigma_N^2}\right)^k L_k^0\left(\frac{y^2}{2\sigma_N^2}\right)\mathcal{E}\left\{|R|^{2k}\right\}, \qquad (13)$$

where $\sigma_N^2$ is the variance of the real and imaginary parts of $\mathbf{N}$, $\mathcal{E}\{\cdot\}$ denotes expectation, and $L_n^\alpha(\cdot)$ is the $n$-th order Laguerre polynomial with parameter $\alpha$, ([22], p. 1037)

$$L_k^\alpha(z) = \sum_{m=0}^{k}\begin{pmatrix} k+\alpha \\ k-m \end{pmatrix}\frac{(-1)^m}{m!}\, z^m. \qquad (14)$$

The Laguerre polynomial $L_n^\alpha(\cdot)$ may be defined recursively as well ([22], p. 1037)

$$L_{n+1}^\alpha(z) = \frac{2n+1+\alpha-z}{n+1}\,L_n^\alpha(z) - \frac{n+\alpha}{n+1}\,L_{n-1}^\alpha(z), \quad (15)$$

where $L_0^\alpha(z) = 0$ and $L_1^\alpha(z) = 1+\alpha-z$. Equation (15) is usually more convenient than (14) in numerical computations. It is interesting to note that, from (13), the *p.d.f* of the detected envelope depends only on the even moments of $R$.

The probability that the detected envelope exceeds some threshold $v$, i.e. the complementary distribution function $\mathcal{F}_{|\mathbf{Y}|}(v)$, may now be obtained from (13), or by using another result of Goldman [21]

$$\mathcal{F}_{|\mathbf{Y}|}(v) = \exp\left(-\frac{v^2}{2\sigma_N^2}\right)\left\{1 - \frac{v^2}{2\sigma_N^2} \cdot \sum_{n=1}^{\infty}\frac{1}{n\,n!}\left(-\frac{1}{2\sigma_N^2}\right)^n L_{n-1}^1\left(\frac{v^2}{2\sigma_N^2}\right)\mathcal{E}\left\{R^{2n}\right\}\right\}. \qquad (16)$$

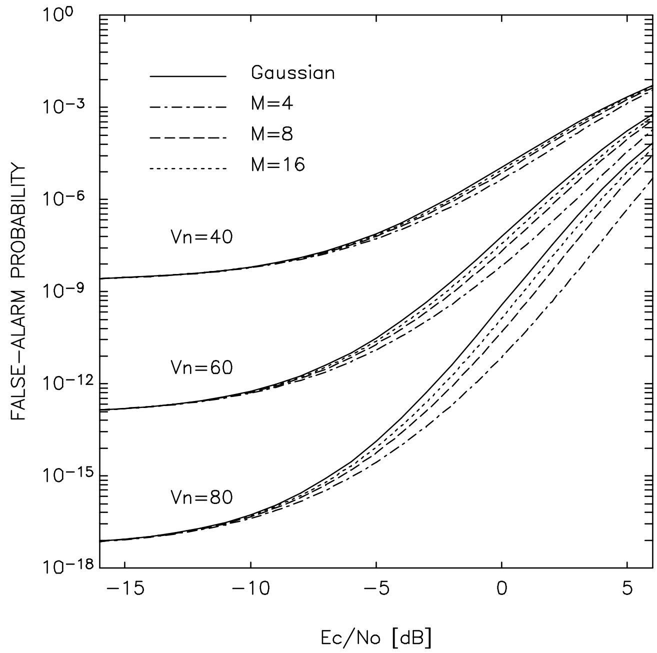


Fig. 3. Exact probability $\mathcal{F}_{|\mathbf{Y}|}(V_n)$ of exceeding the normalized threshold $V_n$ versus per-chip signal-to-noise ratio for various correlation lengths $M$, and the Gaussian approximation; $V_n$ is the normalized threshold; relative clock-offset $\Delta = 0.5$.

Equation (16) is a very convenient starting point for checking the validity of the Gaussian approximation to the binomial distribution of the partial period correlations $\mathcal{R}_M(K, D)$, since it is quite easy to numerically evaluate the moments $\mathcal{E}\{R^{2k}\}$ for any distribution.

In Fig. 3, the resulting $\mathcal{F}_{|\mathbf{Y}|}(v)$ under hypothesis $H_0$ (i.e. false alarm probability) versus the signal-to-noise ratio $E_c/N_0$ is given for $M = 4$, 8 and 16, along with the corresponding results for the Gaussian approximation, for three different values of the normalized detection threshold (see Fig. 1)

$$V_n = \frac{V_T^2}{\sigma_N^2}, \qquad (17)$$

that would be typical for correlation periods of the order of $100\,T_c$. From Fig. 3 we see that for $M$ as low as 16, the error is of the order of a fraction of 1 dB, and it can be shown that exactly the same conclusion holds under hypothesis $H_1$, but the figure is much less illustrative since changing $M$ also changes the mean, as follows from (8).

Furthermore, from Fig. 3 it can be seen that the detection curves converge for large $E_c/N_0$, as should have been expected since large $E_c/N_0$ implies that we are moving away from the tails of the binomial distribution. At first it may be surprising that the curves converge for low $E_c/N_0$ as well, i.e. that the Gaussian approximation holds exceptionally well even for error probabilities as low[3] as $10^{-17}$ (far away in the tails of the binomial distribution). This effect, however, is also expected, since in that region it is the thermal noise that determines the performance; the self-noise term due to $R$ in (2) is negligible in comparison

[3] We note that in practice typical false alarm probabilities would be in the range $10^{-3}$ to, say, $10^{-9}$.

to the thermal noise term $\mathbf{N}$.

As $M$ is typically of the order of at least one-hundred in real applications, we thus conclude that the Gaussian approximation to the correlator self-noise is justified.

## V. Gaussian Approximation

In Section IV it was shown that the distribution of $|\mathbf{Y}|$ depends only on the moments of $R$, whose distribution can be approximated, in all practical applications, by a Gaussian distribution with mean $m_i$, and variance $\sigma_i^2$, as given in (11) and (12) respectively. Among the many possible methods to express the moments of a Gaussian *r.v.* $R$, for the present purpose the most useful representation appears to be

$$\mathcal{E}\left\{R^{2k}\right\} = \left(-\frac{\sigma_i^2}{2}\right)^k H_{2k}\left(j\sqrt{\frac{m_i^2}{2\sigma_i^2}}\right). \tag{18}$$

where $H_n(z)$ is the $n$-th order Hermite polynomial. These polynomials may also be defined recursively ([22], p. 1033) as follows:

$$H_{n+1}(z) = 2z\,H_n(z) - 2n\,H_{n-1}(z), \tag{19}$$

where $H_0(z) = 1$ and $H_1(z) = 2z$. Equation (18) is easily verified by applying the familiar *generating-function* identity ([22], p. 1034)

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} H_n(z)\frac{t^n}{n!} \tag{20}$$

to the characteristic function of a Gaussian *r.v.*. If (18) is now substituted into (13), the *p.d.f.* of $|\mathbf{Y}|$ becomes

$$\mathcal{P}_{|\mathbf{Y}|}(y) = \frac{y}{\sigma_N^2}\exp\left(-\frac{y^2}{2\sigma_N^2}\right)$$

$$\cdot \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{\sigma_i^2}{4\sigma_N^2}\right)^k L_k^0\left(\frac{y^2}{2\sigma_N^2}\right) H_{2k}\left(j\sqrt{\frac{m_i^2}{2\sigma_i^2}}\right), \tag{21}$$

and a similar result can be obtained for the cumulative distribution function by using (16).

For a fixed argument, the Hermite polynomials in (21) quickly increase with $k$ and, due to the imaginary argument, alternate in polarity; the resulting alternating series may be difficult to evaluate. A numerically more convenient result can be obtained from (21) by expanding $L_k^\alpha(\,\cdot\,)$ as in (14). Upon interchanging the order of summations, and by using the following ([23], vol. II, p. 708)

$$\sum_{l=0}^{\infty}\frac{z^l}{l!}H_{2l+n}(x) = (1+4z)^{-(n+1)/2}\exp\left(\frac{4zx^2}{1+4z}\right)$$

$$\cdot\, H_n\left(\sqrt{\frac{x^2}{1+4z}}\right), \tag{22}$$

equation (21) turns into

$$\mathcal{P}_{|\mathbf{Y}|}(y) = \frac{y}{\sigma_N\sqrt{\sigma_i^2+\sigma_N^2}}\exp\left[-\frac{y^2}{2\sigma_N^2} - \frac{m_i^2}{2(\sigma i^2+\sigma_N^2)}\right]$$

$$\cdot \sum_{k=0}^{\infty}\frac{1}{2^k(k!)^2}\left(\frac{y^2}{2\sigma_N^2}\right)^k F_k, \tag{23}$$

where

$$F_k = \left(-\frac{\sigma_i^2}{2(\sigma_i^2+\sigma_N^2)}\right)^k H_{2k}\left(j\sqrt{\frac{m_i^2\sigma_N^2}{2\sigma_i^2(\sigma_i^2+\sigma_N^2)}}\right). \tag{24}$$

With the aid of (18), it may be readily verified that $F_k$ in (24) is the $k$-th moment of an auxiliary Gaussian random variable $R_*$, with mean $m_* = m_i\,\sigma_N/(\sigma_i^2+\sigma_N^2)$ and variance $\sigma_*^2 = \sigma_i^2/(\sigma_i^2+\sigma_N^2)$. It can also be shown that (23), together with (24), is exactly[4] the result obtained previously by Polydoros and Weber [4], without explicitly using the theory of circularly symmetric random variables.

For the present purpose, however, an even more useful result can be obtained from (23) by using the following identity (see Appendix A)

$$\sum_{m=0}^{\infty}\frac{(-1)^m}{(m!)^2}H_{2m}(jb)\,a^{2m} =$$

$$\begin{cases} e^{4a^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^k}\begin{pmatrix}2k\\k\end{pmatrix}\left(\frac{a}{b}\right)^k I_k(4ab), & b\neq 0\\ e^{2a^2}I_0(2a^2), & b=0\end{cases} \tag{25}$$

where $I_k(z)$ is the modified Bessel function of the first kind which, for integral order $k$ can be defined as

$$I_k(z) = \sum_{m=0}^{\infty}\frac{1}{m!\,(m+k)!}\left(\frac{z}{2}\right)^{2m+k}. \tag{26}$$

By substituting (25) (for $b\neq 0$) into (23), and by noticing that the non-zero argument of the Hermite polynomial corresponds to the case $m_i\neq 0$, i.e. to the hypothesis $H_1$, we obtain

$$\mathcal{P}_{|\mathbf{Y}|}(y) = \sqrt{\frac{\sigma_1^2+\sigma_N^2}{\sigma_N^2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}}\begin{pmatrix}2k\\k\end{pmatrix}\left(\frac{\sigma_1^2}{\sigma_N^2}\cdot\frac{y}{m_1}\right)^k$$

$$\cdot\frac{y}{\sigma_1^2+\sigma_N^2}\exp\left[-\frac{y^2+m_1^2}{2(\sigma_1^2+\sigma_N^2)}\right] I_k\left(\frac{ym_1}{\sigma_1^2+\sigma_N^2}\right). \tag{27}$$

In the case $m_i = 0$, i.e. under the hypothesis $H_0$, with the aid of (25) (for $b=0$), (23) simplifies to

$$\mathcal{P}_{|\mathbf{Y}|}(y) = \frac{y}{\sigma_N\sqrt{\sigma_0^2+\sigma_N^2}}$$

$$\cdot\exp\left[-\frac{y^2}{4\sigma_N^2}\,\frac{\sigma_0^2+2\sigma_N^2}{\sigma_0^2+\sigma_N^2}\right] I_0\left(\frac{y^2}{4\sigma_N^2}\,\frac{\sigma_0^2}{\sigma_0^2+\sigma_N^2}\right). \tag{28}$$

It should be noted that (28) coincides with the result from [4], and (27) appears to be new.

As for the complementary distribution function, it is obvious that the last four factors in (27) represent the familiar

[4]In particular, this can be verified if in [4] equation (A.7) is combined with (A.23), and by noting that in the line following (A.23) the definition of the parameter $m_n$ has a typographical-error and should read as $m_n = m_1\sigma_N/(1+\rho_1^2)$.

Rician density of order $(2k+2)$. Since the complementary distribution of the even-order Rician density can be expressed in terms of the *generalized* Marcum's $Q$-function, thus for hypothesis $H_1$

$$\mathcal{F}_{|\mathbf{Y}|}(v) = \sqrt{\frac{\sigma_1^2+\sigma_N^2}{\sigma_N^2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}} \left( \begin{array}{c} 2k \\ k \end{array} \right) \left( \frac{\sigma_1^2}{\sigma_N^2} \right)^k \cdot Q_{k+1}\left( \sqrt{\frac{m_1^2}{\sigma_1^2+\sigma_N^2}}, \sqrt{\frac{v^2}{\sigma_1^2+\sigma_N^2}} \right), \quad (29)$$

where

$$Q_k(x,y) = Q(x,y) + e^{-\frac{x^2+y^2}{2}} \sum_{n=1}^{k-1} \left(\frac{y}{x}\right)^k I_k(xy), \quad (30)$$

and $Q(x,y)$ is the familiar Marcum's $Q$-function [24].

A similar series expansion is possible for $\mathcal{F}_{|\mathbf{Y}|}(v)$ in the $H_0$ case, but instead we choose to give bounds. In Appendix B it is shown that for hypothesis $H_0$

$$\frac{1}{1+\sigma_0^2/2\sigma_N^2} \cdot f_0(v) \le \mathcal{F}_{|\mathbf{Y}|}(v) \le f_0(v), \quad (31)$$

where $f_0(v)$ is given by

$$f_0(v) = \sqrt{\frac{\sigma_0^2+\sigma_N^2}{\sigma_N^2}} \cdot \exp\left[ -\frac{v^2}{4\sigma_N^2} \frac{\sigma_0^2+2\sigma_N^2}{\sigma_0^2+\sigma_N^2} \right] I_0\left( \frac{v^2}{4\sigma_N^2} \frac{\sigma_0^2}{\sigma_0^2+\sigma_N^2} \right). \quad (32)$$

Equations (29), and (31) together with (32), are still too complex for most practical applications. Further simplifications are possible if we note that in practice both the mean $m_1$ and the (unnormalized) detection threshold $V_T$ are of the same order of magnitude, and that both have to be much larger than the total variance $\sigma_i^2+\sigma_N^2$ in order to have reliable detection. Now, since the modified Bessel function $I_k(z)$ for large arguments very quickly converges to the first term of its asymptotic expansion, independently of the order we have ([22], p. 962)

$$I_k(z) \approx \frac{1}{\sqrt{2\pi z}} e^z, \ (z \gg 1), \quad (33)$$

and using the well-known asymptotic behavior of Marcum's $Q$-function ([24], pp. 585–589)

$$Q(x,y) \approx \frac{1}{\sqrt{2\pi(y-x)}} e^{-\frac{(x-y)^2}{2}}, \ (xy \gg 1, \ y \gg y-x), \quad (34)$$

then, after some simple algebra it can be shown that the second term on the right hand side of (30) can be neglected asymptotically, i.e. that

$$Q_k(x,y) \approx Q(x,y), \ (xy \gg 1, \ y \gg y-x). \quad (35)$$

Employing the result ([23], vol. I, p. 711)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k}} \left( \begin{array}{c} 2k \\ k \end{array} \right) z^k = (1+z)^{-1/2}, \quad (36)$$

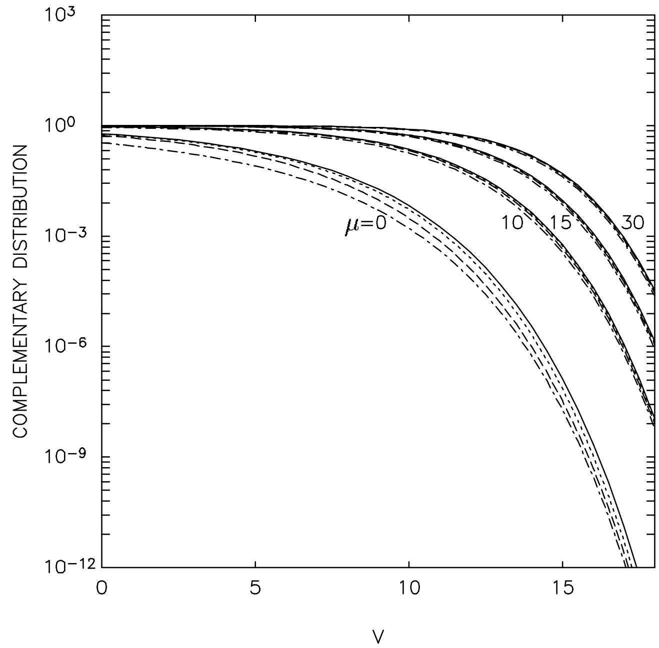


Fig. 4. Exact cumulative distribution function $\mathcal{F}_{|\mathbf{Y}|}(V)$ versus the normalized variable $V = v/(\sigma_i^2+\sigma_N^2)$ for $\sigma_i^2/\sigma_N^2 = 0.1$ (dotted), 0.5 (dashed) and 10 (dash-dot line); the solid line is the approximation from (37) and (38) ; the normalized mean of the distribution is $\mu = m_i^2/(\sigma_i^2+\sigma_N^2)$.

from (29) and (35) we arrive at the following asymptotically tight approximation for $\mathcal{F}_{|\mathbf{Y}|}(v)$ under hypothesis $H_1$ (i.e. for the detection probability)

$$P_D \approx Q\left( \sqrt{\frac{m_1^2}{\sigma_1^2+\sigma_N^2}}, \sqrt{\frac{V_T^2}{\sigma_1^2+\sigma_N^2}} \right). \quad (37)$$

We note that (37) can be used in almost all practical systems, because in most cases we would have $m_1$ and $V_T$ of the same order of magnitude, and $m_1, V_T \gg \sigma_1^2+\sigma_N^2$.

Under the hypothesis $H_0$, we recognize from (31) that $f_0(v)$ is an asymptotic expansion for $\mathcal{F}_{|\mathbf{Y}|}(v)$ . For $V_T \to \infty$, further simplifications are possible if we substitute (33) into (32), resulting in the simpler asymptotically tight expression

$$P_{FA} \approx \exp\left( -\frac{V_T^2}{2(\sigma_0^2+\sigma_N^2)} \right). \quad (38)$$

For finite $V_T/\sigma_N$, however, accuracy of the approximation (38) depends also on the ratio $\sigma_0^2/\sigma_N^2$. This is illustrated in Fig. 4, where we see that the accuracy decreases with increasing $\sigma_0^2/\sigma_N^2$. Nevertheless, in the region of interest (false alarm probability smaller than, say, $10^{-3}$), even for $\sigma_0^2/\sigma_N^2 = 10$, the difference would be less than 1dB. Also, from the same figure we see that the approximation (37) holds very well for all system parameters of practical interest, as expected. Thus, if $\sigma_0^2$ and $\sigma_N^2$ are of the same order of magnitude, we can freely use the approximations (37) and (38) for the detection and false alarm probability.

From (3) and (12), which define $\sigma_0^2$ and $\sigma_N^2$, it is easy to see that the condition $\sigma_0^2/\sigma_N^2 \gg 1$ corresponds to very high signal-to-noise ratios ($E_c/N_0$), not likely to be encountered in practice, so that asymptotical expressions (37) and

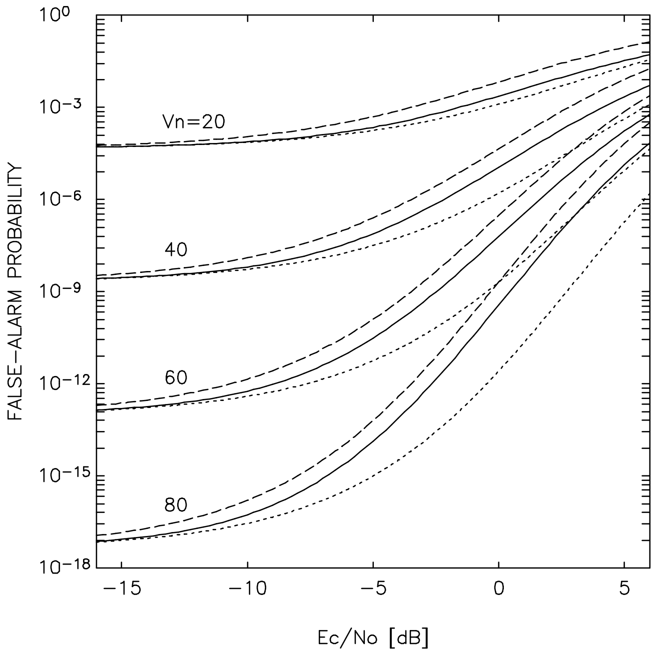


Fig. 5. Exact false alarm probability $P_{FA}$ (solid), approximations from equation (38) (dashed) and from reference [4] (dotted line) versus per-chip signal-to-noise ratio for $M = 128$, $\Delta = 0.5$, non-coherent correlator; $V_n$ is the normalized threshold from (17) .

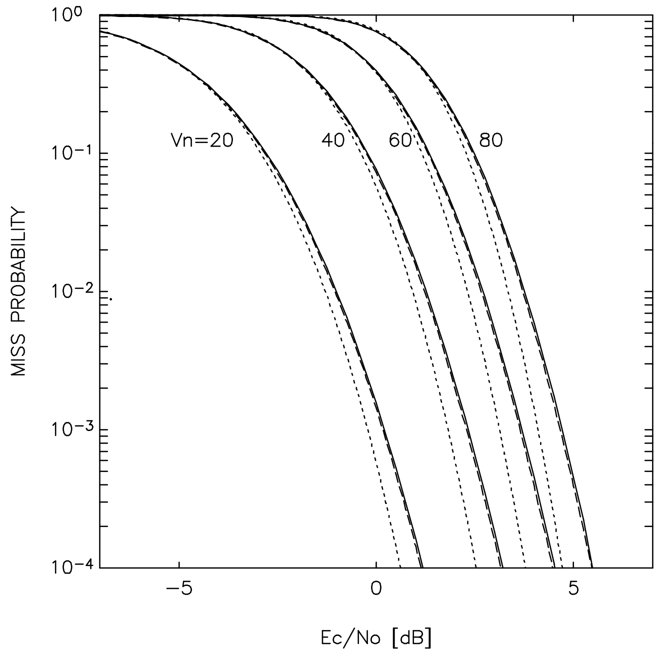


Fig. 6. Exact miss probability $1 - P_D$ (solid), approximations from equation (37) (dashed) and from reference [4] (dotted line) versus per-chip signal-to-noise ratio for $M = 128$, $\Delta = 0.5$, non-coherent correlator; $V_n$ is the normalized threshold from (17).

(38) should hold quite well under almost all sets of system parameters.

Furthermore, since $Q(0, v) = \exp(-v^2/2)$, we see that (37) and (38) can be combined into the single expression

$$\mathcal{F}_{|\mathbf{Y}|}(v) \approx Q\left(\sqrt{\frac{m_i^2}{\sigma_i^2 + \sigma_N^2}}, \sqrt{\frac{v^2}{\sigma_i^2 + \sigma_N^2}}\right). \tag{39}$$

Equation (39) clearly indicates that, in the non-coherent correlation of BPSK-DS waveforms, the distribution of the detected envelope can be very well approximated by the distribution of the signal embedded in the equivalent noise that is a sum of the thermal noise $\sigma_N^2$ and the correlator self-noise $\sigma_i^2$. Depending on the hypothesis $H_i, (i = 0, 1)$, the self-noise variance $\sigma_i^2$ is given in (12).

Finally, in Fig. 5 and Fig. 6 we show the exact detection performance of the non-coherent correlator that integrates $M = 128$ chips, along with the approximation (39) and a somewhat intuitive approximation proposed in [4], where the equivalent noise process was approximated by the sum $\sigma_N^2 + \sigma_i^2/2$. It can be seen that the approximation (39) holds very well for all detection probabilities, outperforming the approximation from [4] especially in the most interesting range of false alarm probability ($10^{-7}$ to $10^{-3}$), and detection probabilities (larger than 0.9).

The approximation from [4], however, holds amazingly well, especially for lower normalized thresholds $V_n$. Due to the asymptotic nature of (39), for $V_n \approx 30$ both approximations give roughly the same error. Bellow that, for instance at $V_n = 20$ (corresponding to the normalized threshold that could be typical for e.g. $M = 64$, which also appears to have been the "state-of-the-art" length of many passive correlators throughout much of the last decade[5] ), the approximation from [4] is more exact for evaluation of the false alarm probability.

## VI. Conclusions

In this paper, the performance of a spread spectrum non-coherent correlation detector for BPSK signaling in an AWGN environment is analyzed. Average performance over the ensemble of random code sequences is considered, leading to the binomial distribution of the partial period autocorrelations.

By using the theory of circularly symmetric random variables, exact results for the binomial distribution were obtained, as well as for the frequently adopted Gaussian approximation to the binomial distribution. It has been shown that the use of the Gaussian approximation is completely justified in the present application. Under this assumption, exact results for the detection performance of the non-coherent correlator were derived. In some special cases, it has been shown that these are identical to the ones obtained previously by Polydoros and Weber [4].

Since the exact results appear to be quite complex, simple asymptotically tight approximations were obtained, which are also very tight under most sets of system parameters likely to appear in practice. These approximations show that the detection performance of the non-coherent correlator is determined by the equivalent Gaussian noise process, whose variance equals the sum of the variances due to the thermal noise and the correlator self-noise. Compared to the approximations developed previously [4], the derived approximate expressions were generally found to perform

[5] E.g. digital programmable 64-tap model TDC1023 by "TRW" [13].

better.

This paper addressed only the performance of the BPSK acquisition detector; i.e., the detection and false alarm probabilities were evaluated as a function of the detection time and physical parameters of the channel. By using these results, higher level aspects of the performance of an acquisition system such as detection/verification logic, search strategy, quantization of the code phase uncertainty region, *a priori* probability distribution of the code phase, etc., can be analyzed following the methods described in [7; 8; 9], either in terms of the distribution of the acquisition time, or its moments.

## Appendix A

In order to obtain (25), we introduce the following identity

$$\frac{1}{(m!)^2} = 2^{2m}\frac{1}{(2m)!}\frac{1}{\pi}\int_0^{\pi}\cos^{2m}\theta\, d\theta, \qquad \text{(A.1)}$$

which can be proved starting from the expression for the integral $\int_0^{\pi}\cos^{2m}\theta\, d\theta$ ([23], vol. I, p. 171).

Thus, the expression on the left hand side of (25) becomes

$$\frac{1}{\pi}\int_0^{\pi}\sum_{n=0}^{\infty}\frac{(2aj\cos\theta)^n}{n!}H_n(jb)\, d\theta. \qquad \text{(A.2)}$$

The summation in (A.2) can be evaluated with the aid of (20) to yield

$$\frac{1}{\pi}\int_0^{\pi}e^{-4ab\cos\theta+4a^2\cos^2\theta}\, d\theta. \qquad \text{(A.3)}$$

Now, if $b=0$, (A.3) can be written as

$$\frac{1}{\pi}\, e^{2a^2}\int_0^{\pi}e^{-2a^2\cos\alpha}\, d\alpha, \qquad \text{(A.4)}$$

and since the modified Bessel function of integer order can be defined also as ([22], p. 958)

$$I_k(z) = \frac{(z/2)^k}{\pi 2^{-2k}}\,\frac{k!}{(2k)!}\int_0^{\pi}e^{z\cos\phi}\sin^{2k}\phi\, d\phi, \qquad \text{(A.5)}$$

from (A.4) and (A.5) equation (25) for $b=0$ immediately follows.

For the case $b \neq 0$, by expanding the exponential function into a power series, (A.3) can be transformed into

$$\frac{1}{\pi}e^{4a^2}\int_0^{\pi}e^{-4ab\cos\theta}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(4a^2\sin^2\theta)^k\, d\theta, \qquad \text{(A.6)}$$

and further into

$$\frac{1}{\pi}e^{4a^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}(4a^2)^k\int_0^{\pi}e^{-4ab\cos\theta}\sin^{2k}\theta\, d\theta. \qquad \text{(A.7)}$$

The integral in (A.7) may now be expressed in terms of Bessel functions using (A.5) to obtain (25).

It should be added that this derivation was inspired in part by [25].

## Appendix B

With the aid of (28), the complementary distribution function $\mathcal{F}_{|\mathbf{Y}|}(v)$ under hypothesis $H_0$ can be transformed into the integral

$$\mathcal{F}_{|\mathbf{Y}|}(v) = c\int_{v^2}^{\infty}e^{-au}I_0(bu)\, du, \qquad \text{(B.1)}$$

where

$$\begin{aligned} a &= \frac{1}{4\sigma_N^2}+\frac{1}{4(\sigma_0^2+\sigma_N^2)} \\ b &= \frac{1}{4\sigma_N^2}-\frac{1}{4(\sigma_0^2+\sigma_N^2)} \\ c &= \left[4\sigma_N^2(\sigma_0^2+\sigma_N^2)\right]^{-1/2}. \end{aligned} \qquad \text{(B.2)}$$

If $I_0(z)$ is expanded into a power-series following (26), we arrive at

$$\mathcal{F}_{|\mathbf{Y}|}(v) = c\sum_{k=0}^{\infty}\frac{(b/2)^{2k}}{k!\,k!}\int_{v^2}^{\infty}u^{2k}e^{-au}\, du. \qquad \text{(B.3)}$$

The integral in (B.3) is, by definition, the incomplete Gamma-function ([22], pp.317) and can be expressed as ([22], pp.941)

$$\int_{v^2}^{\infty}u^{2k}e^{-au}\, du = e^{-av^2}\sum_{j=0}^{2k}\frac{(2k)!}{j!}\cdot\frac{v^{2j}}{a^{2k-j+1}}, \qquad \text{(B.4)}$$

and if this is substituted into (B.3), it turns into

$$\mathcal{F}_{|\mathbf{Y}|}(v) = \frac{c}{a}e^{-av^2}\sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{b}{2a}\right)^{2k}\sum_{j=0}^{2k}\frac{1}{j!}(av^2)^j. \qquad \text{(B.5)}$$

Now, (B.5) can be lower-bounded if only the term for $j=2k$ is retained in the second sum, so that

$$\mathcal{F}_{|\mathbf{Y}|}(v) \geq \frac{c}{a}e^{-av^2}\sum_{k=0}^{\infty}\frac{1}{(k!)^2}\left(\frac{b\,v^2}{2}\right)^{2k} \qquad \text{(B.6)}$$

The sum on the right hand side of (B.6), following (26), is the expansion for $I_0(bv^2)$. Thus

$$\mathcal{F}_{|\mathbf{Y}|}(v) \geq \frac{c}{a}e^{-av^2}I_0(bv^2), \qquad \text{(B.7)}$$

and if $a$, $b$ and $c$ from (B.3) are now substituted into (B.7), the lower bound in (31) follows.

The upper bound in (31) can also be derived from (B.1), by partial integration. Details are given in ([4], pp. 559).

## References


[1] M. K. Simon, J. K. Omura, R. A. Scholtz, and B. Levitt, *Spread-Spectrum Communications*, Rockville, MD: Computer Science Press, 1985.

[2] F. Hemmati and D. L. Schilling, "Upper bounds on the partial correlation of PN sequences", *IEEE Trans. Commun.*, vol. COM-31, pp. 917–922, July 1983.

[3] S. Davidovici, L. B. Milstein, and D. L. Schilling, "A new rapid acquisition technique for direct sequence spread-spectrum communications", *IEEE Trans. Commun.*, vol. COM-32, pp. 1161–1168, Nov. 1984.

[4] A. Polydoros and C. L. Weber, "A unified approach to serial-search spread-spectrum code acquisition-Part II: A matched-filter receiver", *IEEE Trans. Commun.*, vol. COM-32, pp. 550–560, May 1984.

[5] J. S. Lehnert and M. B. Pursley, "Error probabilities for binary direct-sequence spread-spectrum communications with random signature sequences", *IEEE Trans. Commun.*, vol. COM-35, pp. 87–98, Jan. 1987.

[6] E. S. Sousa "Interference modeling in a direct sequence spread-spectrum packet radio networks", *IEEE Trans. Commun.*, vol. COM-38, pp. 1475–1482, 1990.

[7] V. M. Jovanović, "Analysis of strategies for serial-search spread-spectrum code acquisition–direct approach", *IEEE Trans. Commun.*, vol. COM-36, pp. 1208–1220, Nov. 1988.

[8] V. M. Jovanović, "On the distribution function of the spread-spectrum cod acquisition time", *IEEE J. Selec. Areas Commun.*, vol. SAC-10, pp. 760–769, May 1992.

[9] A. Polydoros and C. L. Weber, "A unified approach to serial-search spread-spectrum code acquisition-Part I: general theory", *IEEE Trans. Commun.*, vol. COM-32, pp. 542–549, May 1984.

[10] D. P. Morgan, J. M. Hannah and J. H. Collins, "Spread-spectrum synchronizer using a SAW convolver and recirculation loop", *Proc. IEEE*, vol. 64, pp. 751–753, May 1976.

[11] E. G. Magill, D. M. Grieco, R. Dyck, and P. C. Y. Chen, "Charge-coupled device pseudo-noise matched filter design", *Proc. IEEE*, vol. 67, pp. 50–60, Jan. 1979.

[12] G. L. Turin, "An introduction to digital matched filters", *Proc. IEEE*, vol. 64, pp. 1092–1112, July 1976.

[13] S. S. Rappaport and D. M. Grieco "Spread-spectrum acquisition: methods and technology", *IEEE Commun. Mag.*, pp. 6–21, June 1984.

[14] E. W. Seiss and C. L. Weber, "Acquisition of direct sequence signals with modulation and jamming", *J. Selec. Areas. Commun.*, vol. SAC-4, pp. 254–272, Mar. 1986.

[15] S. W. Golomb, *Shift-Register Sequences*, San Francisco: Holden Day, 1967.

[16] N. E. Bekir, R. A. Scholtz, and L. R. Welch, "Partial-period correlation properties of PN sequences", in *Proc. NTC*, Birmingham, Alabama, pp. 35.1.1–35.1.4, Dec. 1978.

[17] J. K. Jao and M. Elbaum, "First-order statistics of a non-Rayleigh fading signal and its detection", *Proc. IEEE*, vol. 66, pp. 781–789, July 1978.

[18] J. S. Bird and D. A. George, "The use of the Fourier-Bessel series in calculating error probabilities for digital communication systems", *IEEE Trans. Commun.*, vol. COM-29, pp. 1357–1365, Sep. 1981.

[19] W. R. Bennett, "Distribution of the sum of randomly phased components", *Quatr. J. Appl. Math.*, vol. 5, pp. 385–393, 1948.

[20] R. Esposito and L. R. Wilson, "Statistical properties of two sine waves in Gaussian noise", *IEEE Trans. Inform. Theory*, vol. IT-19, pp. 176–183, Mar. 1973.

[21] J. R. Goldman, "Statistical properties of of a sum of sinusoids and Gaussian noise and its generalization to higher dimensions", *Bell Syst. Tech. J.*, vol. 53, pp. 557–580, Apr. 1974.

[22] I. S. Gradshtein and I. M. Ryzhik, *Tables of Integrals, Series and Products*, New York: Academic, 1980.

[23] A. P. Prudnikov, Y. A. Brychkov and O.I.Marichev, *Integrals and Series*, London, UK: Gordon and Breach Publ., 1988.

[24] M. Schwartz, W. R. Bennett, and S. Stein, *Communication System and Techniques*, New York: McGraw – Hill, 1966.

[25] H. N. Sagon, "An integral relevant to the detection of a signal in a polarized Gaussian noise", *IEEE Trans. Inform. Theory*, vol. IT-16, p. 232, Mar. 1970.

**Vladan M. Jovanović** (S'82-M'85) was born in Belgrade, Serbia, Yugoslavia. He received the B.S., M.S., and Ph.D. degrees in electrical engineering in 1981, 1985, and 1988 respectively, all from the University of Belgrade, Yugoslavia. In 1981, he joined IMTEL - Institute of Microwave Techniques and Electronics, Belgrade, where he led several research and development projects in the field of spread spectrum. From May 1991 to February 1993 he was a research associate at the University of Toronto, Toronto, Canada, where he was involved in studies of spread spectrum synchronization and multipath fading channels. He is now engaged in RF technology planning with Bell Mobility Cellular, Etobicoke, Canada. He has published articles in the areas of spread spectrum, error correction coding, effecient modulation, synchronization, and fading channel characterization.

**Elvino S. Sousa** (S'80 - M'86) was born in Graciosa, Azores, (Portugal) on December 28, 1956. He received the B.A.Sc. degree in engineering science, and the M.A.Sc. degree in electrical engineering from the University of Toronto in 1980 and 1982 respectively, and the Ph.D. degree in electrical engineering from the University of Southern California in 1985. Since 1986 he has been with the department of Electrical and Computer Engineering at the University of Toronto where he is presently an Associate Professor. Since 1986 he has been a Natural Sciences and Engineering Research Council of Canada (NSERC) University Research Fellow.

He has performed research in the areas of packet radio networks, spread spectrum systems, mobile communications, and indoor wireless communications. At the University of Toronto he has taught graduate courses in error-correcting codes and mobile communications. He has given various lectures and short courses in mobile communications throughout the world. He is the Technical Program Chairman for the Sixth IEEE International Symposium on Personal Indoor and Mobile Radio Communications (PIMRC95).